COMMENTS TO THE ARTICLE «LONG AND SHORT MEMORY IN ECONOMICS: FRACTIONAL-ORDER DIFFERENCE AND DIFFERENTIATION»

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Abstract: *in these comments, we give the corrected equations for exact fractional differences, which are written with misprints in the article «Long and short memory in economics: fractional-order difference and differentiation» (IRA-International Journal of Management and Social Sciences. 2016. Vol. 5. No. 2. P. 327-334.). The exact fractional differences can be considered as exact discrete analogues of the Liouville fractional derivatives of integer and non-integer order. These fractional differences and derivatives of non-integer order can be used to describe economic processes with power-law fading memory.*
Keywords: *long memory, short memory, economic processes with memory, ARIMA model, ARFIMA model, exact differences, fractional difference, Grunwald-Letnikov differences, fractional derivative, exact discretization.*

КОММЕНТАРИИ К СТАТЬЕ «LONG AND SHORT MEMORY IN ECONOMICS: FRACTIONAL-ORDER DIFFERENCE AND DIFFERENTIATION»

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Аннотация: *в данных комментариях приводятся исправленные формулы для точных конечных разностей, которые в статье «Long and short memory in economics: fractional-order difference and differentiation» (IRA-International Journal of Management and Social Sciences, 2016. Vol. 5. No. 2. P. 327-334.) приведены с опечатками. Точные дробные разности можно рассматривать как точные дискретные аналоги дробных производных Лиувилля целого и нецелого порядка. Эти дробные разности и производные нецелого порядка могут быть использованы для описания экономических процессов с степенной угасающей памятью.*
Ключевые слова: *долговременная память, кратковременная память, экономические процессы с памятью, модель ARIMA, модель ARFIMA, точные разности, дробные разности, разности Грюнвальда-Летникова, дробная производная, точная дискретизация.*

In the paper [1, p. 332], equations (21) and (23) of the exact fractional differences are given with misprints. These comments provide corrected formulas for the exact fractional differences.

The exact discretization of the derivatives of integer and non-integer orders [2, 3] and the corresponding exact finite differences [2, 3] were initially proposed in [4, 5, 6, 7] as derivatives on lattices. In economics, they were used in [8, 9]. Derivatives of non-integer order and the corresponding fractional finite differences allow us to describe economic processes with power-law dynamic memory [10].

The exact fractional differences $\Delta^{\alpha}_{T,exact}$ of order α are defined by the equation

$$\Delta^{\alpha}_{T,exact}\, Y(t) \coloneqq \sum_{m=-\infty}^{\infty} K_{\alpha}(m) \cdot Y(t - m \cdot T), \qquad (1)$$

where $K_{\alpha}(m)$ is the kernel of the exact fractional difference of the form

$$K_{\alpha}(m) = \cos\left(\frac{\pi\alpha}{2}\right) \cdot K^{+}_{\alpha}(m) + \sin\left(\frac{\pi\alpha}{2}\right) \cdot K^{-}_{\alpha}(m). \qquad (2)$$

The kernels $K^{+}_{\alpha}(m)$ and $K^{-}_{\alpha}(m)$ are given by the expressions

$$K^{+}_{\alpha}(m) \coloneqq \frac{\pi^{\alpha}}{\alpha+1} F_{1,2}\left(\frac{\alpha+1}{2}; \frac{1}{2}, \frac{\alpha+3}{2}; -\frac{\pi^2 \cdot m^2}{4}\right), (\alpha > -1), \qquad (3)$$

$$K_{\alpha}^{-}(m) := -\frac{\pi^{\alpha}\cdot m}{\alpha+2} F_{1,2}\left(\frac{\alpha+2}{2}; \frac{3}{2}, \frac{\alpha+4}{2}; -\frac{\pi^2\cdot m^2}{4}\right), (\alpha > -2), \qquad (4)$$

where we use the generalized hypergeometric function

$$F_{1,2}(a; b, c; z) := \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\cdot\Gamma(b)\cdot\Gamma(c)}{\Gamma(a)\cdot\Gamma(b+k)\cdot\Gamma(c+k)} \cdot \frac{z^k}{k!}. \qquad (5)$$

Using equation (5), the kernel (2) can be written in the form

$$K_{\alpha}(m) = \sum_{k=0}^{\infty} \frac{(-1)^k\cdot\pi^{2k+\alpha+\frac{1}{2}}\cdot m^{2k}}{2^{2k}\cdot k!\cdot\Gamma\left(k+\frac{1}{2}\right)} \cdot \left(\frac{\cos\left(\frac{\pi\alpha}{2}\right)}{\alpha+2k+1} - \frac{\pi\cdot m\cdot\sin\left(\frac{\pi\alpha}{2}\right)}{(\alpha+2k+2)\cdot(2k+1)}\right). \qquad (6)$$

For α<0 expression (2) with the kernel (6) defines the discrete fractional integration [2, 3].

For the arbitrary positive integer order α=n, the kernel $K_{\alpha}(m)$ of the exact difference can be represented by the equation

$$K_n(m) = \sum_{k=0}^{\left[\frac{n+1}{2}\right]+1} \frac{(-1)^{m+k} n!\pi^{n-2k-1}}{(n-2k)!\cdot m^{2k+2}} \left((n-2k)\cos\left(\frac{\pi n}{2}\right) + \pi m \sin\left(\frac{\pi n}{2}\right)\right) \qquad (7)$$

for $m \neq 0$, and for $m = 0$ the kernel is written by the expression

$$K_n(0) = \frac{\pi^n}{n+1} \cdot \cos\left(\frac{\pi n}{2}\right). \qquad (8)$$

The exact finite difference (1) of the first order (α=1) is defined by the equation

$$\Delta_{T,exact}^{1} Y(t) := \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \cdot (Y(t - T\cdot m) - Y(t + T\cdot m)), \qquad (9)$$

where the sum implies the Cesaro or Poisson-Abel summation [3, p. 55-56; 13]. An important characteristic property of the exact finite difference (9) is the Leibniz rule (the product rule) in the form

$$\Delta_{T,exact}^{1}\left(X(t)\cdot Y(t)\right) = \left(\Delta_{T,exact}^{1} X(t)\right)\cdot Y(t) + X(t)\cdot\left(\Delta_{T,exact}^{1} Y(t)\right) \qquad (10)$$

which is satisfied for all $X(t), Y(t)$ from the space of entire functions. Exact finite difference of second and next integer orders can be derived by the recurrence formulas $\Delta_{T,exact}^{k+1} Y(t) = \Delta_{T,exact}^{1}\left(\Delta_{T,exact}^{k} Y(t)\right)$.

In the paper [1, p. 332], equations (21) and (23) should be replaced by equations (4) and (6), respectively.

It should be noted that the exact fractional differences are exact discrete analogues of the Liouville fractional derivatives. Equations of discrete macroeconomic models, which are used the exact finite differences, are exact discrete analogs of differential equations of models with continuous time for a wide class of solutions (for example, see [9] and [10, 11, 12, 14]).

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