On the solvability of the Cauchy problem for a singularly perturbed integro-differential equations in partial derivatives of the first order with a turning point

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О разрешимости задачи Коши сингулярно-возмущенных интегро-дифференциальных уравнений в частных производных первого порядка с точкой поворота

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Abstract: *in this study we investigated the solvability of the Cauchy problem solution and its structure for a singularly perturbed integro-differential equations with a turning point derivatives. In solutions found an integral representation.*

Аннотация: *в работе изучена разрешимость решений задачи Коши и ее структура для сингулярно–возмущенных интегро-дифференциальных уравнений в частных производных с точкой поворота. В решении найдено интегральное представление.*

Keywords: *integral equation, partial differential equation of first order, the principle of contraction mappings, Lipschitz condition, nonlinearity.*

Ключевые слова: *интегральное уравнение, дифференциальное уравнение в частных производных первого порядка, принцип сжатых отображений, условие Липшица, нелинейность.*

The essence of the proposed method is converting solutions, finding the solutions of the initial transformation of the Cauchy problem for a singularly perturbed integro-differential equations in partial derivatives with a turning point, and bringing to the Voltaire equivalent integral equation of II kind. Here are algebraically-functional bases of conversion method making the theory of differential, integral equations.

In many problems of analytical and asymptotic theory of differential and integral equations applied the method of converting solutions. For example, in Paper [1] Chapter VIII is devoted the method of converting solutions, allowing to integrate a predetermined differential equation or explore the properties of its solutions.

In this regard, we introduce a definition. Let's Ω - a set, the operators A and K represent it in himself. Consider the equation

$$Ax = b \qquad (1)$$

where b - a fixed element of Ω and conversion

$$x = Ky \qquad (2)$$

From (1), (2) directly have

$$AKy = b. \qquad (3)$$

Hence, if there is a "semi-inverse" operator $(AK)^{-1}$ to the operator of AK, we obtain

$$y = (AK)^{-1} b \qquad (4)$$

and from (3), (2) we have solution of equation (1) in the form

$$x = K(AK)^{-1} b \qquad (5)$$

Definition1. The operator K will be called the operator of converting solutions of A.

NOTE1. If (2) has the form

$$x = Kx,$$

then (1) can be written as

$$x = (AK)^{-1} b,$$

in particular, if it appears $AK = E$, E -the unit operator, $K = A^{-1}$.

NOTE 2. It is necessary to choose the operator K, so as to obtain a simplified new operator equation (3), to which it would be possible to apply one of the following methods:

- topological methods for proving the existence of solutions, for example, the principle of compressed mappings;

- methods expansions of solutions, for example, methods of making the expansion in power series (the first Lyapunov method);

- directly produce various, including asymptotic evaluation using assumptions with respect to A and K and the element b, the independent variable tends to a limiting value.

NOTE 3. We note that the definition 1 includes the methods of integral transforms $y = Fx$, Fourier transforms (Laplace), substituting (2) in the form

$y = Fx$ where $F = K^{-1}$,

l.e pre-suppose the existence of the inverse operator K^{-1}, whereas in (4) assumes the existence of semi-inverse operator $(AK)^{-1}$.

Now consider the singularity perturbed integro-differential equation in partial derivatives with a turning point

$$\varepsilon\left(\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\right)+\sin nt\,u(t,x)= f(t,x,u(t,x))+\int_0^t K\left(t,s,u(s,x)\right)ds \qquad (5)$$

with the initial condition

$$u(0,x)=\varphi(x). \qquad (6)$$

Here are mathematical notation used in this paper:

R-number line $R_+ := (0;+\infty)$;

$\bar{C}^{\alpha,\beta\cdots}(\Omega\to\Lambda)$- space of functions bounded and continuous together with its derivatives to the corresponding order;

$Lip(L\,|_u)$ - class of functions satisfying a Lipschitz condition and with coefficient L.

Assumption (T). Let $n\in N$ - fixed number,

$$f(t,x,u)\in\bar{C}\left(\left[0,T\right]\times R\times R\right)\cap Lip\left(L_1\big|_u\right),\ \varphi(x)\in\bar{C}^1(R),$$

$$K(t,\tau,u)\in\bar{C}\left(\left(0\le\tau\le t\le T\right)\times R\right)\cap Lip\left(L_2\big|_u\right).$$

The solution of the Cauchy problem (1) - (2) in the form

$$u(t,x)=\varphi(x-t)+\int_0^t e^{-\frac{\alpha}{\varepsilon}(t-s)+\frac{\beta s}{\varepsilon}}\frac{1}{\varepsilon}Q(s,x-t+s)ds\,,$$

Where $Q(t,x)$- a new unknown function to be determined; $\alpha,\beta\in R_+$ and their values will be determined later.

Successively differentiating with respect to t and x relation (7), we have

$$u_t(t,x)=-\varphi'(x-t)+\frac{1}{\varepsilon}e^{\frac{\beta t}{\varepsilon}}Q(t,x)-\frac{\alpha}{\varepsilon}(u-\varphi)-\int_0^t e^{-\frac{\alpha}{\varepsilon}(t-s)+\frac{\beta}{\varepsilon}s}\frac{1}{\varepsilon}Q_x(s,x-t+s)ds; \qquad (8)$$

$$u_x(t,x)=\varphi'(x-t)+\int_0^t e^{-\frac{\alpha}{\varepsilon}(t-s)+\frac{\beta s}{\varepsilon}}\frac{1}{\varepsilon}Q_x(s,x-t+s)ds. \qquad (9)$$

From whence

$$u_t(t,x)+u_x(t,x)=\frac{1}{\varepsilon}e^{\frac{\beta t}{\varepsilon}}Q(t,x)-\frac{\alpha}{\varepsilon}u+\frac{\alpha}{\varepsilon}\varphi(x-t).$$

Multiplying both sides of this equation by ε, we have

$$\varepsilon(u_t+u_x)+\alpha u(t,x)=e^{\frac{\beta t}{\varepsilon}}Q(t,x)+\alpha\varphi(x-t). \qquad (10)$$

In view of (6)

$$\varepsilon(u_t+u_x)+\sin t\,u=e^{\frac{\beta t}{\varepsilon}}Q(t,x)-\left(\alpha-\sin t\right)u(t,x)+\alpha\varphi(x-t)$$

Where, taking into account (7), we obtain

$$Q(t,x)=e^{-\frac{\beta t}{\varepsilon}}\left[f(t,x,u)+\int_0^t K(t,\tau,u(t,\tau))d\tau\right]+$$

$$+e^{-\frac{\beta t}{\varepsilon}}(\alpha-\sin nt)\left[\varphi(x-t)+\int_0^t e^{-\frac{\alpha}{\varepsilon}(t-s)+\frac{\beta s}{\varepsilon}}\frac{1}{\varepsilon}Q(s,x-t+s)ds\right]-e^{-\frac{\beta t}{\varepsilon}}\alpha\varphi(x-t),$$

or

$$Q(t,x)=e^{-\frac{\beta t}{\varepsilon}}f(t,x,\varphi(x-t)+\int_0^t e^{-\frac{\alpha}{\varepsilon}(t-s)+\frac{\beta s}{\varepsilon}}\frac{1}{\varepsilon}Q(s,x-t+s)ds+$$

$$+e^{-\frac{\beta t}{\varepsilon}}\int_0^t\left[K(t,s,\varphi(x-\tau)+\int_0^\tau e^{-\frac{\alpha}{\varepsilon}(\tau-s)+\frac{\beta s}{\varepsilon}}\frac{1}{\varepsilon}Q(s,x-\tau+s)ds\right]d\tau+ \qquad (11)$$

$$+e^{-\frac{\beta t}{\varepsilon}}(\alpha-\sin nt)\int_0^t e^{-\frac{\alpha}{\varepsilon}(t-s)+\frac{\beta s}{\varepsilon}}\frac{1}{\varepsilon}Q(s,x-t+s)ds+e^{-\frac{\beta t}{\varepsilon}}\sin nt\,\varphi(x-t)\equiv P[Q].$$

Then, to prove the existence and uniqueness of solution of the Cauchy problem (5) - (6) to the Voltaire nonlinear integral equation of II kind apply the contraction mapping principle, l. e to the operator equation

$$u=Pu\,,$$

where the operator Pu - right side of equation (11).

Let the set

$$\Omega=\{Q(t,x):Q(t,x)\in\overline{C}^{(1,1)}([0,T]\times R)\cup\|Q\|\le h\}$$

Values T and h will be determined later.

From (11), the assumption (T), we have

$$\|PQ\|\le e^{-\frac{\beta T}{\varepsilon}}\left[M_1+M_2T\right]+(\alpha+1)\int_0^t e^{-\frac{(\alpha+\beta)}{\varepsilon}(t-s)}\frac{1}{\varepsilon}\|Q(s,x-t+s)\|ds+e^{-\frac{\beta t}{\varepsilon}}M_3\le$$

$$\le e^{-\frac{\beta t}{\varepsilon}}(M_1+M_2T+M_3)+\frac{\alpha+1}{\alpha+\beta}\|Q\|.$$

Hence, by the definition of Ω, we have

$$\|PQ\|\le e^{-\frac{\beta T}{\varepsilon}}(M_1+M_2T+M_3)+\frac{\alpha+1}{\alpha+\beta}h.$$

If we choose T, α,β,h so that

$$e^{-\frac{\beta T}{\varepsilon}}(M_1+M_2T+M_3)+\frac{\alpha+1}{\alpha+\beta}h\le h, \qquad (12)$$

the operator $P[Q]$ puts the set of $P[Q]:\ \Omega\to\Omega$

Now we show that the operator P is a contraction operator. From (11) using the assumption (T), we obtain

$$\|P[Q_1]-P[Q_2]\|\le\Bigg\|e^{-\frac{\beta t}{\varepsilon}}[f(t,x,\varphi(x-t)+\int_0^t e^{-\frac{\alpha}{\varepsilon}(t-s)+\frac{\beta s}{\varepsilon}}\frac{1}{\varepsilon}Q_1(s,x-t+s)ds)-$$

$$-f(t,x,\varphi(x-t)+\int_0^t e^{-\frac{\alpha}{\varepsilon}(t-s)+\frac{\beta s}{\varepsilon}}\frac{1}{\varepsilon}Q_2(s,x-t+s)ds)\Bigg]\Bigg\|+$$

$$+\left\|e^{-\frac{\beta t}{\varepsilon}}\left[\int_0^t K\left(t,s,\varphi(x-\tau)+\int_0^\tau e^{-\frac{\alpha}{\varepsilon}(\tau-s)+\frac{\beta s}{\varepsilon}}\frac{1}{\varepsilon}Q_1(s,x-\tau+s)ds\right)-\int_0^t K\left(t,s,\varphi(x-\tau)+\int_0^\tau e^{-\frac{\alpha}{\varepsilon}(\tau-s)+\frac{\beta s}{\varepsilon}}\frac{1}{\varepsilon}Q_2(s,x-\tau+s)ds\right)\right]\right\|+$$

$$+\left\|e^{-\frac{\beta t}{\varepsilon}}(\alpha-\sin nt)\int_0^t e^{-\frac{\alpha}{\varepsilon}(t-s)+\frac{\beta s}{\varepsilon}}\frac{1}{\varepsilon}[Q_1(s,x-t+s)-Q_2(s,x-t+s)]ds\right\|\le\left[\frac{L_1+L_2+(\alpha+1)}{\alpha+\beta}\right]\|Q_1(t,x)-Q_2(t,x)\|,$$

where it is considered that $\|\sin nt\| \le 1, \quad \forall n \in N$.

Now we impose on α, β the following restrictions:

$$\frac{L_1 + L_2 + (\alpha+1)}{\alpha+\beta} < 1. \qquad (13)$$

Then (11) implies that operator $P[Q]$ is a compression operator on the set Ω. On the principle of contraction mapping equation (11) has a unique solution $Q(t,x) \in \Omega$. Substituting the function found in (7), we obtain the solution of the Cauchy problem (5)-(6).

Now we investigate the differential properties of solutions of the Cauchy problem (5)-(6) in the region Ω. For all $Q(t,x) \in \Omega$ from the equation (7) implies the inequality

$$\|u(t,x)\| \le \|\varphi(x-t)\| + \left\| \int_0^t e^{-\frac{\alpha}{\varepsilon}(t-s)+\frac{\beta s}{\varepsilon}} \frac{1}{\varepsilon} Q(s, x-t+s)ds \right\| \le K_\varphi + e^{-\frac{\alpha}{\varepsilon}T} \frac{h}{\alpha+\beta} - K_0 = const.$$

From (8) we have

$$\|u_t(t,x)\| = \left\| -\varphi'(x-t) + \frac{1}{\varepsilon} e^{\frac{\beta t}{\varepsilon}} Q(t,x) - \frac{\alpha}{\varepsilon}(u-\varphi) - \int_0^t e^{-\frac{\alpha}{\varepsilon}(t-s)+\frac{\beta}{\varepsilon}s} \frac{1}{\varepsilon} Q_x(s, x-t+s)ds \right\| \le \|-\varphi'(x-t)\| +$$

$$+ \left\| \frac{1}{\varepsilon} e^{\frac{\beta t}{\varepsilon}} Q(t,x) \right\| + \left\| -\frac{\alpha}{\varepsilon}(u-\varphi) \right\| + \left\| -\int_0^t e^{-\frac{\alpha}{\varepsilon}(t-s)+\frac{\beta}{\varepsilon}s} \frac{1}{\varepsilon} Q_x(s, x-t+s)ds \right\| \le N_1 - const.$$

From (9) we have

$$\|u_x(t,x)\| = \|\varphi'(x-t)\| + \left\| \int_0^t e^{-\frac{\alpha}{\varepsilon}(t-s)+\frac{\beta s}{\varepsilon}} \frac{1}{\varepsilon} Q_x(s, x-t+s)ds \right\| \le N_2 - const.$$

Thus we have proved that all the derivatives included in the equation (5) are uniformly bounded. Thus, we have

THEOREM. Let the assumptions (T), (12), (13). Then $\exists T_0 > 0$ such that the Cauchy problem (5) - (6) has a solution $u(t,x) \in \overline{C}^{(1,1)}([0,T_0] \times R)$, which has a representation in the form of the integral (7).

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